Problem Set III: Due at last class

N.B.: - Only 5 problems from this set need be submitted. However, you are responsible for ***all*** on the Final Exam.
- There will be a Set IV.

1a.) Using the same type of consideration as discussed in class, show that the decorrelation rate for a particle in an ensemble of stochastic electric fields in 1D scales as:

$$1/\tau_c \sim \left(k^2 D_{QL}\right)^{1/3}.$$

Here, D_{QL} is the quasilinear diffusion co-efficient and decorrelation is defined relative to a wave number k.

b.) Extend the idea of integration over unperturbed orbits from 218A to show that formally the response of the distribution function to the electric field is given by

$$f_{\substack{k\\ \omega}} = -\frac{q}{m} E_{\substack{k\\ \omega}} \partial\langle f\rangle/\partial v \int_0^\infty d\tau\, e^{i(\omega - kv)\tau} e^{ik\delta x(-\tau)},$$

where $\delta x(-\tau)$ is the deviation from the unperturbed obit.

c.) Now, assume $\delta x(-\tau)$ is produced by diffusion in velocity and thus show

$$f_{\substack{k\\ \omega}} = -\frac{q}{m} E_{\substack{k\\ \omega}} \frac{\partial\langle f\rangle}{\partial v} \int_0^\infty d\tau\, e^{i(\omega - kv)\tau} e^{-k^2 D\tau^3/3}.$$

d.) What is the physics of the effect discussed in c.)?

e.) Taking $\tau^3/\tau_c^3 \rightarrow \tau/\tau_c$ for convenience, what does this problem imply about resonance widths at finite amplitude? What does validity of the quasilinear calculation of D imply about the resonance width?

2a.) Consider a passive scalar, with concentration c, immersed in a turbulent flow. c obeys the equation:

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla} c - D\nabla^2 c = \tilde{f}_c .$$

Let c have dissipation rate α, i.e.

$$\alpha = \tilde{c}_0^2 v_0 / \ell_0 .$$

i.) Calculate the $K41$ inertial range spectrum for concentration fluctuations.

ii.) *Quantitatively* discuss what happens if

$$D << \nu \text{ (viscosity)}, \; D >> \nu .$$

b.) Consider low Mach number incompressible turbulence, with $M = v_0/c_s << 1$. Show

$$\ell_{diss}/\ell_{mfp} \sim M^{-1} Re^{1/4} .$$

Thus, the validity of continuum hydrodynamics gets *better* at high *Re*.

3.) Consider a bounded, 2D shear flow with $V_y = V_y(x)$, with no slip boundary conditions.

a.) Derive the linear equation governing the inviscid stability of this flow. (Hint: consider vorticity dynamics.)

b.) Show that an inflection point (i.e. a point where $V_y''(x) = 0$) is necessary for instability. (Hint: Assume growth, and construct a complex quadratic form. What does this form imply about the growth rate?)

c.) Derive the quasilinear equation for mean vorticity ω for this problem. When is this applicable?

d.) Now, assume that fluctuations are maintained by external stirring. What can be said about the mean time asymptotic vorticity profile assuming that no slip boundary conditions apply at each boundary? Discuss the physics of your result. (Hint: Review the H-theorem for quasilinear theory.)

4.) This problem extends Problem 3.

Now consider the flow of Problem 3 in an MHD fluid with $\eta = \nu$, but negligible. Assume a magnetic field $\underline{B} = B_0\hat{y}$ is present.

a.) Derive the linear eigenmode equation for the ideal MHD problem. Discuss the structure of the problem.

b.) What is the effect of the magnetic field on linear stability? Quantify your answer.

c.) What is the relation between vorticity flux $\langle \tilde{v}_x \tilde{\omega} \rangle$ and Reynolds stress $\langle \tilde{v}_x \tilde{v}_y \rangle$? Explain the physics behind this result.

d.) Derive the mean vorticity and mean flow evolution equations. What is the effect of the magnetic field on momentum transport?

e.) Derive the quasilinear fluctuation-driven resistivity.

5.) Now consider the system of Problem 4 *in 3D*, with $\eta > \nu$, so the inductive electric field is negligible. Assume the velocity field is forced on large scale.

a.) Derive the mean fluctuation energy balance equation for this system. What is the effect of the magnetic field?

b.) Will an inertial range cascade be possible, in general?

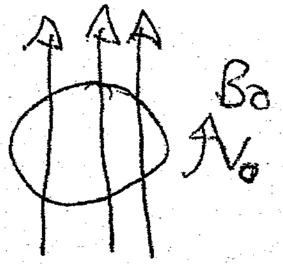
c.) Discuss the ultimate effect of the field on the turbulent flow field. For a strong field, what type of eddy structure would you expect?

6a.) Show that hydrodynamic helicity $H = \int d^3x\, \underline{v} \cdot \underline{\omega}$ is an inviscid invariant of an unmagnetized fluid. Assume $\underline{\nabla} \cdot \underline{v} = 0$.

b.) What is this physical meaning of H? What properties would you expect of a flow with high relative helicity (i.e. $H \big/ \left[\langle v^2 \rangle \langle \omega^2 \rangle \right]^{1/2}$). Would this flow be turbulent?

c.) Derive the hydrodynamic helicity balance for a forced, viscous flow. What are the sources and sinks? Compare to the magnetic helicity budget.

7.) What is the width of the magnetic boundary layer formed at the boundary of an eddy when magnetic field is expelled, in the case shown below (discussed in class).



b.) Will flux expulsion *always* occur? Estimate the conditions under which it will *not*. Discuss the physics of your result.

8.) Consider the separation between two 'test' particles, attached to an ensemble of frozen field lines, with a fluctuation spectrum as predicted by the Goldreich-Sridhar model. Assume critical balance.

a.) Calculate the rate of separation of the neighboring particles, as they travel in the mean field ($\hat{z}$) direction. (Hint: Think Richardson, but for field lines.)

b.) How does the rate of separation compare to that of test particles in Navier-Stokes turbulence? What is the reason for this result?

9.) Consider a rapidly rotating incompressible fluid. Take $\underline{\Omega} = \Omega_0 \hat{z}$. Obviously, the Coriolis effect is crucial here.

a.) Derive the dispersion relation for inertial waves, with $\omega \sim \Omega$. Take $\nu = 0$. What is the physics of these waves? What is the relation between their group and phase velocity?

b.) Now take $\underline{B}_0 = B_0 \hat{z}$, as well and consider $\omega << \Omega$. For $\nu = \eta = 0$, show that magnetostrophic waves with

$$\omega = \pm \frac{1}{4\pi\rho_0} \left(\underline{k} \cdot \underline{B}_0 \right)^2 k / 2 \left(\underline{\Omega} \cdot \underline{k} \right)$$

exist for $\left(\underline{k} \cdot \Omega \right)^2 >> \left(\underline{k} \cdot \underline{B}_0 \right)^2 k^2 / 4\pi\rho_0$. These waves are called magnetostrophic waves.

10.) Derive the Reduced MHD Equations, for incompressible, strongly magnetized MHD dynamics in a (nearly) uniform $\underline{B}$ field.

a.) Do this by extending the derivation of 2D MHD, discussed previously.

b.) Do this by starting from $E_{\parallel} = 0$ and $\underline{\nabla} \cdot \underline{J} = 0$, where $\underline{J}_{\perp}$ includes $\underline{E} \times \underline{B}$ and parallel currents.

c.) Read the notes on MHD and the H.R. Strauss '76 paper in Phys. Fluids. From an orderings perspective, what is the key to Reduced MHD?